INVESTIGATION OF THE NONSTATIONARY CONDITIONS OF FORMATION OF AN OPTICAL FIBER. II. PERTURBATION OF THE FEED AND DRAW RATES

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An algorithm for studying the nonstationary conditions of formation of an optical fiber is studied and the results of numerical calculations with perturbation of the feed and draw rates are presented.

The problem of the stability of the drawing process, studied in part one [1], determines the region of parameters where continuous formation of an optical fiber (OF) is possible, i.e., small perturbations of the intitial state decay in time and do not lead to decomposition of the liquid jet. To optimize commerical production of optical fibers the character of the transient processes accompanying different stepped perturbations and the sensitivity of the diameter of an OF to small fluctuations of the drawing parameters near the stationary values are of great interest. The solution of this problem will make possible the following:

optimization of the control system with respect to the output diameter of the fiber;

finding the frequencies at which the reaction of the drawing process to external perturbations is maximum (this information is necessary in order to optimize the drawing equipment, for example, to eliminate vibrations at resonance frequencies, to determine the optimal tolerances in the geometric dimensions of the preform to stabilize the furnace temperature, etc.); and,

determining the region of parameters in which the drawing process is least sensitive to external perturbations.

1. Formulation of the Problem. The study of the reaction to external perturbations is based on a mathematical model [2, 3] describing the conditions of formation of fibers by the molding method with heating of the preform in a furnace. The transfer function was calculated both based on the solution of the starting nonlinear system of equations [3] and based on its linear analog [4]. In the first case the steady-state values of R(x), V(x), and T(x) were taken as the initial data and a stepped perturbation of one or another parameter was given at the time $\tau = 0$. Using the linearized system of equations [4] to find the transfer function the energy equation can be written in the following form:

$$\frac{\partial \tilde{T}}{\partial \tau}=\frac{\lambda}{\text{Pe}}\frac{\partial^2 \tilde{T}}{\partial x^2}+\varphi_3(x)\frac{\partial \tilde{T}}{\partial x}+\varphi_4(x)\tilde{T}+\alpha_3(x)\frac{\partial \tilde{R}}{\partial x}+\varphi_4(x)\tilde{R}+\beta_3(x)\tilde{V}+\varphi_5(x)\tilde{\text{St}}+\varphi_6(x)\tilde{T}_p, \quad (1)$$

since, unlike the study of the stability of the process, in this case it is necessary to study perturbations of the number St and the temperature of the furnace, which analogously to $\overline{R}$, $\overline{V}$, and $\overline{T}$ are represented in the form of expansions $\overline{\text{St}}(\tau)=\text{St}[1+\tilde{\text{St}}(\tau)]$, $\overline{T}_p(\eta,\tau)=T_p(\eta)[1+\tilde{T}_p(\eta,\tau)]$. Here

$$\varphi_5(x)=-\frac{2(1+R'^2)^{1/2}}{RT}(T-T_0)\,\text{St},$$

$$\varphi_6(x)=\frac{16\chi R_p(R_p-R)\varepsilon_p}{RT}\int_0^l\frac{\beta T_p^4[R_p-R+kR'(x-\eta)]}{[(x-\eta)^2+(R_p-R)^2]^2}d\eta.$$

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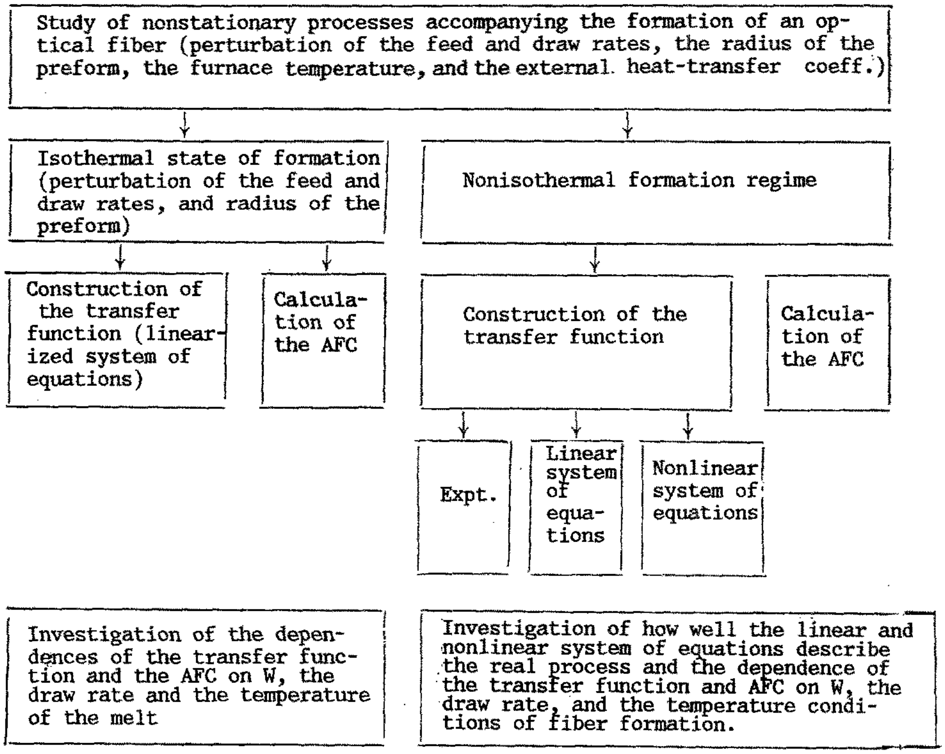

Fig. 1. The structure of the algorithm employed to study numerically the nonstationary process accompanying the formation of OF.

The rest of the notation in Eq. (1) corresponds to [4]. The boundary conditions for the linearized system of equations are obtained by substituting the expansions of $\bar{R}$, $\bar{V}$, and $\bar{T}$ into the boundary conditions [3] and assume the following form:

$$\tilde{V}(x, \tau) = \tilde{V}_0(\tau), \quad \tilde{R}(x, \tau) = \tilde{R}_0(\tau), \quad T_k(x, \tau) = T_k(\tau) \quad \text{at} \quad x = 0, \tag{2}$$

$$\tilde{V}(x, \tau) = \tilde{V}_b(\tau), \quad \frac{\partial \tilde{T}(x, \tau)}{\partial x} = 0 \quad \text{at} \quad x = 1, \tag{3}$$

where $\tilde{V}_0(\tau)$, $\tilde{R}_0(\tau)$, $\tilde{T}_k(\tau)$ and $\tilde{V}_b(\tau)$, are, generally speaking, arbitrary functions of the time and describe the perturbation of the boundary parameters; in particular, in the study of transient characteristics they are step functions.

The linearized initial- and boundary-value problem was solved numerically using the method of fractional steps [5]. In each time layer the system was split according to the physical processes (R, V, T) into three fractional steps [6]. The starting relations were approximated by an implicit difference scheme with second-order accuracy, and the system of linear algebraic equations obtained was solved by the method of flux factorization [7].

The ultimate purpose of studying the reaction of the OF drawing process to external disturbances is to find the amplitude-frequency characteristics (AFCs) A_ω, which are the modulus of the frequency transfer function $\tilde{W}(i\omega)$, where ω is the circular frequency (a real number). Currently there exist two efficient algorithms for constructing the AFC of the OF drawing process with different external perturbations.

The first approach, which was employed in [8-10] for the case of perturbation of the drawing velocity and the radius of the preform, is based on the assumption that an arbitrary perturbation can be represented as a superposition of harmonic functions of time (expanded in a Fourier series [11]), so that the problem of finding the AFC is completely solved by finding the solution for harmonic disturbances of the form

$$\tilde{\mathrm{St}} = \delta e^{-i\omega\tau}, \quad \tilde{T}_p = \delta e^{-i\omega\tau}, \quad \tilde{R}_0 = \delta e^{-i\omega\tau}, \quad \tilde{V}_b = \delta e^{-i\omega\tau}, \quad \tilde{V}_0 = \delta e^{-i\omega\tau}, \tag{4}$$

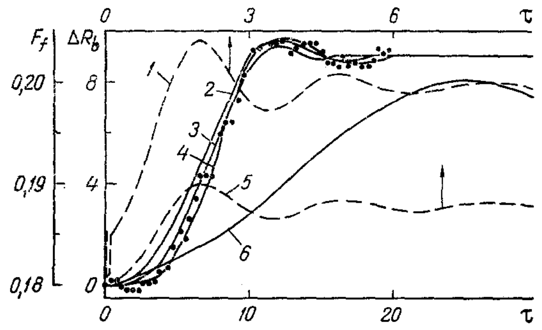

Fig. 2. Measurement of the OF radius with stepped perturbation of the feed rate (solution of the nonlinear system of equations): 1) perturbation of F_f for 5; 2) experimental values T_p = 2320°C, R_p = 0.015 m, F_f = 0.025 N, R_0 = 0.0046 m, R_b = 62.5 μm, V_b = 3 m/sec, the magnitude of the perturbation is equal to 15%; 3) T_{p2} = 2350°C, ξ = 0.565, R_p = 0.015 m, F_f = 0.019 N, R_0 = 0.0046 m, R_b = 62.5 μm, V_b = 3 m/sec, the magnitude of the perturbation is equal to 15%; 4) experimental values, 2220; 0.015; 0.075; 0.046; 62.5; 3; 15; 5) 2250; 0.375; 0.01; 0.186; 0.005; 64; 10; 10; 6) 2250; 0.375; 0.01; 0.011; 0.005; 64; 0.5; 20 (the dots are the experimental values for 4). F_f, N; ΔR_b, μm, τ, sec.

where δ is the amplitude (a real number).

If there is a periodic perturbation at the input to the system $y \in \{\tilde{\text{S}}\text{t}, \tilde{T}_P, \tilde{V}_g, \tilde{R}_0, \tilde{V}_b\}$, then in the stationary state, because the equations are linear, there will also be a harmonic function of time with the same frequency at the outlet [12], so that the solution has the form

$$\tilde{R} = \delta(r + i\tilde{r})e^{-i\omega\tau}, \quad \tilde{V} = \delta(v + i\tilde{v})e^{-i\omega\tau}, \quad \tilde{T} = \delta(t + i\tilde{t})e^{-i\omega\tau}. \tag{5}$$

Substituting Eq. (5) and separating the real and imaginary parts transforms the starting equations into a system of second-order ordinary differential equations:

$$\frac{3\mu}{\text{Re}} v'' + \left(\beta_1 - \frac{\alpha_1}{2}\right) v' + \beta_2 v = \omega\tilde{v} - \alpha_2 r + \frac{\omega\alpha_1}{V}\tilde{r} - \varphi_1 t' - \varphi_2 t, \tag{6}$$

$$\frac{3\mu}{\text{Re}} \tilde{v}'' + \left(\beta_1 - \frac{\alpha_1}{2}\right) \tilde{v}' + \beta_2 \tilde{v} = -\omega v - \frac{\omega\alpha_1}{V} r - \alpha_2\tilde{r} - \varphi_1\tilde{t}' - \varphi_2\tilde{t}, \tag{7}$$

$$\frac{\lambda}{\text{Pe}} t'' + \varphi_3 t' + \varphi_4 t = \omega\tilde{t} - \alpha_4 r + \frac{R\alpha_3}{2V} v' - \beta_3 v + \frac{\omega\alpha_3}{V}\tilde{r} - k_1\varphi_5 - k_2\varphi_6, \tag{8}$$

$$\frac{\lambda}{\text{Pe}} \tilde{t}'' + \varphi_3\tilde{t}' + \varphi_4\tilde{t} = -\omega t - \frac{\omega\alpha_3}{V} r - \alpha_4\tilde{r} + \frac{R\alpha_3}{2V}\tilde{v}' - \beta_3\tilde{v}, \tag{9}$$

$$r' = -\frac{\omega}{V}\tilde{r} - \frac{1}{2}v', \tag{10}$$

$$\tilde{r}' = \frac{\omega}{V} r - \frac{1}{2}\tilde{v}', \tag{11}$$

where the coefficients k_1 and k_2 are weighting factors: k_1 = 1 in the case of perturbation of the furnace temperature, k_2 = 1 for the case of perturbation of the St number, and in the remaining cases $k_1 = k_2 = 0$.

The boundary conditions for Eqs. (6)-(11) are obtained after Eqs. (4) and (5) are substituted into Eqs. (2) and (3), and they will be studied in detail when the problem is formulated for a specific case.

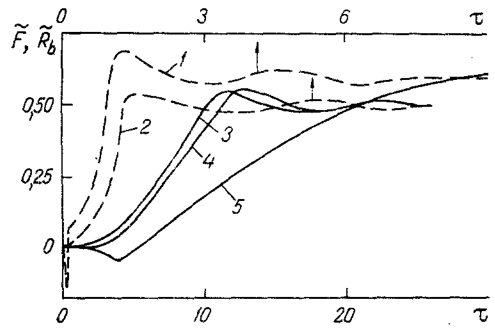

Fig. 3. The change in the radius of the fiber with a stepped perturbation of the draw rate (the solution of the linearized system of equations, $\tilde{V}_0$ = 1%): 1) perturbation of the pulling force for 2; 2) T_{p2} = 2250°C, V_b = 10 m/sec; 3) 2250; 3; 4) 2150; 3; 5) 2250; 0.5. $\tilde{F}$, $\tilde{R}_b$, %.

If the radius of the fiber as a function of time is regarded as the "output," then by definition the AFC is the ratio of the modulus of the amplitude of the perturbation of the cross section of the fiber to the modulus of the amplitude of the perturbation and, according to Eqs. (4) and (5), it can be represented as follows:

$$A_\omega = [r^2(1) + \tilde{r}^2(1)]^{1/2}.$$

To find r and $\tilde{r}$ with x = 1 the system of second-order differential equations (6)-(11) was reduced with the help of the substitution of variables ($u_j \in \{v, \tilde{v}, t, \tilde{t}\}$) to a system of ordinary differential equations, which were solved by the "aiming" method [13] since the values of the functions sought are given at both ends of the region of computation. The integration "forward" (in the direction of increasing argument) and "backward" (in the opposite direction) were performed by the method of Adams [14], after which the solutions were joined at the center of the interval using the algorithm described in [13].

The second approach, which was first used in [15] in order to calculate the AFC of the OF drawing process, is based on the assumptions of the theory of automatic regulation, according to which the frequency transfer function for a linear system is the Fourier transform of its weighting function [16]:

$$\tilde{W}(i\omega) = \int_0^\infty \omega_1(\tau)\, e^{-i\omega\tau} d\tau, \quad (12)$$

where $\omega_1(\tau)$ is the response to a δ-function perturbation. Let $U_1(\omega)$ and $U_2(\omega)$ be the real and imaginary components of $\tilde{W}(i\omega)$; then $A_\omega = [U_1^2(\omega) + U_2^2(\omega)]^{1/2}$. The weighting function can be obtained by differentiating the transfer function with respect to the time $\omega_1(\tau) = d\omega_2(\tau)/d\tau$.

The numerical study of the nonstationary processes accompanying the formation of an OF was performed for perturbations of the feed and draw rates, the radius of the preform, the furnace temperature, and the external heat transfer coefficient. The calculations were performed in two stages using the scheme shown in Fig. 1. The isothermal regime was employed as a model problem which was used to debug the algorithm for calculating the transfer function and the AFC.

In comparing the results of the numerical and experimental study of the OF drawing process (the measurements were performed on the OPTEX drawing machine) the experimental data were smoothed using cubic splines [17] and direct calculations were performed for the following initial data: T_{p1} = 1500°C, ξ = 0.375, R_p = 0.01 m, R_0 = 0.005 m, R_b = 64 μm, St = 2.5, θ_0 = 0.11 m, the physical properties of the glass were taken from [18].

In calculating $\omega_2(\tau)$ the perturbation of the tensile force for the nonlinear system of determining equations was calculated in parallel using the formula [3]

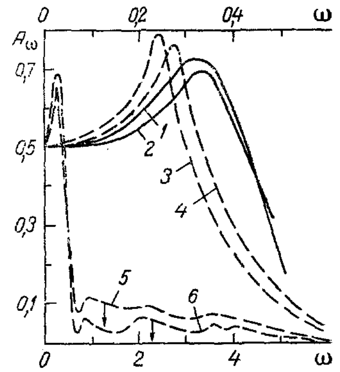

Fig. 4. The AFC of the nonisothermal regime of fiber formation (perturbation of the feed rate): 1) T_{p2} = 2250°C, V_b = 0.5 m/sec; 2) 2250; 3; 3) 2250; 0.5; 4) 2250; 3; 5) 2250; 10; 6) 2150; 10 (solid lines – the solution of Eqs. (6)-(11), dashed lines – Fourier transform of the weighing function; ω is the number of oscillations in 2π sec).

$$F = \frac{3\mu R^2}{\mathrm{Re}}\frac{\partial V}{\partial x} + \frac{R}{\mathrm{We}} - \frac{1}{\mathrm{Fr}}\int_x^1 R^2 dx. \tag{13}$$

Here $F = F_f/\pi\rho l^2 V_0^2$ is the dimensionless pulling force. The expression for calculating the perturbation F for the linearized system of equations is obtained after substituting into Eq. (13) the relations for $\tilde{R}$, $\tilde{V}$, $\tilde{T}$ [1], $\overline{F}(\tau) = F[1 + \tilde{F}(\tau)]$ and linearizing with respect to the variables $\tilde{R}$, $\tilde{V}$, and $\tilde{T}$:

$$\tilde{F} = \frac{1}{F}\left[\left(\frac{R}{\mathrm{We}} + \frac{6\mu R^2 V'}{\mathrm{Re}}\right)\tilde{R} - \frac{2}{\mathrm{Fr}}\int_x^1 R^2\tilde{R}dx + \frac{3\mu R^2 V}{\mathrm{Re}}\frac{\partial \tilde{V}}{\partial x} + \frac{3\mu R V'}{\mathrm{Re}}\tilde{V} - \frac{3\mu a_2 T R^2 V'}{\mathrm{Re}}\tilde{T}\right].$$

2. Perturbation of the Preform Feed Rate. The boundary conditions for calculating the transfer function are obvious, and therefore here and below they will not be studied. We stress that in studying the perturbations of the feed and draw rates and the radius of the preform it is assumed that $\tilde{S}t = \tilde{T}_p = 0$ in Eq. (1). In calculating the AFC based on the solution of Eqs. (6)-(11) substitution of variables ($z_j = du_j/dx$, j = 1, 2, 3, 4) gives a system of ordinary differential equations, in which some of the initial conditions are given at the left end of the working interval ($\tilde{v}$, v, $\tilde{t}$, t, $\tilde{r}$, r) while the other conditions are given at the right end (v, $\tilde{v}$, z_3, z_4). In order to be able to integrate it "forward" and "backwards" to realize the method of "aiming" it is necessary to evaluate the values of the functions sought that are not given at x = 0 and x = 1. By virtue of Eq. (4), $\tilde{v}(0)$, v(0), $\tilde{t}(0)$, t(0), $\tilde{r}(0)$, v(1), $\tilde{v}(1)$ are equal to either unity or zero. For this reason the assumption that $z_j(0) = z_1(1) = z_2(1) = 0$, j = 1, 2, 3, 4 is obvious. For any type of perturbation the exact values of t(1), $\tilde{t}(1)$, r(1), $\tilde{r}(1)$, cannot be given, so that in all calculations their zeroth approximation was employed and then refined by the method of "aiming."

Figures 2 and 3 show the results of the numerical calculation of the transfer function. It follows from these results that the reaction of the fiber drawing process with perturbation of the feed rate is very close to linear (compare Figs. 2 and 3). The latter is also confirmed by the fact that $\omega_2(\tau)$ of the system with negative perturbation, when the conditions for fiber formation are identical, is the same as the mirror image of the curves in Figs. 2 and 3 downwards. As the temperature of the melt is increased the amplitude of the first harmonic $\omega_2(\tau)$ decreases in the region of deformation, while an increase in the draw rate substantially reduces the time of the transient process and reduces the amplitude of oscillations of the fiber radius. With a draw rate of ~10 m/sec the transient process becomes

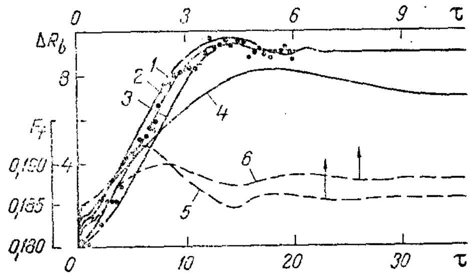

Fig. 5. The change in the OF radius with a stepped perturbation of the draw rate (solution of the nonlinear system of equations): 1) T_{p2} = 2350°C, ξ = 0.565, R_p = 0.015 m, F_f = 0.019 N, R_0 = 0.0046 m, R_b = 62.5 μm, V_b = 3 m/sec, the magnitude of the perturbation is 15%; 2) experimental values T_p = 2320°C, R_p = 0.015 m, F_f = 0.025 N, R_0 = 0.0046 m, R_b = 62.5 μm, V_b = 3 m/sec, the magnitude of the perturbation is 15%; 3) experimental values 2220; 0.015; 0.075; 0.0046; 62.5; 3; −15; 4) 2250; 0.375; 0.01; 0.01; 0.011; 0.005; 64; 0.5; −20; 6) 2250; 0.375; 0.01; 0.186; 0.005; 64; 10; −10; 5) the perturbation of the pulling force F_f for 6 (the dots are the experimental values for 2).

aperiodic, i.e., a practically smooth transition from one state of equilibrium to another is observed. Calculation of the perturbation of the pulling force (Figs. 2 and 3) showed that initially it makes a sharp jump and then starts to undergo synchronous oscillations with $\omega_2(\tau)$, leading them somewhat in phase.

The results of the calculation of the AFC based on the solution of Eqs. (6)-(11) and the Fourier transform of the weighting function are virtually identical, but from the viewpoint of computer time it is best to calculated A_ω using the formula (12) (the computing time is approximately two to three times shorter). In addition, in solving Eqs. (6)-(11) in some cases, when V_b < 3 m/sec, numerical instability is observed (the iteration based on the aiming method does not converge). The AFC has a distinct maximum, whose magnitude is determined by the temperature of the melt and the value of W, while the position on the frequency scale is determined by the draw rate (Fig. 4), and in addition as V_b increases the AFC shifts into the high-frequency region and the amplitude of the oscillations decreases. For a high draw rate increasing the temperature of the melt results in a decrease in the magnitude of the first peak of the AFC (curves 5 and 6 in Fig. 4), but in this case the amplitude of the high-frequency oscillations increases substantially.

3. Perturbation of the Fiber Draw Rate. Figures 5-7 show the results of calculations of the transfer function and the AFC. The basic behavior of these characteristics, noted above for the perturbation of the feed rate, are valid for this case. Here it should be noted that the linear approximation with perturbation of V_b is not as good as for V_0 (see for comparison Figs. 5 and 6). This largely pertains to the starting section of the transfer function (τ > 2 sec), where over a very short time interval a practically jump-like change occurs in the diameter of the fiber (Fig. 5, curves 1, 4, and 5). The most significant difference between the results of the experimental study (curves 2 and 3 in Fig. 5) and the calculation (curve 1 in Fig. 5) is also observed in the section with τ < 2 sec and is evidently caused by the fact that a finite time is required to reduce the draw rate by 15%.

The first peak in the AFC is less distinct than for the perturbation of V_0, and the magnitude of the peak approaches zero as the draw rate increases. In addition, this graph confirms even more graphically the need for choosing correctly the temperature for formation of the OF, since overheating of the melt with a high draw rate results in the possibility of the appearance of high-frequency oscillations of the fiber radius (curves 5 and 6 in Fig. 7).

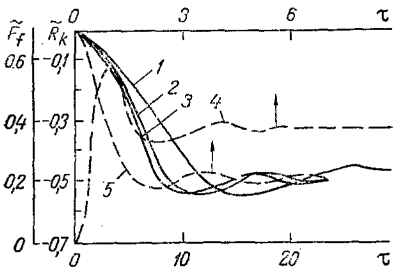

Fig. 6

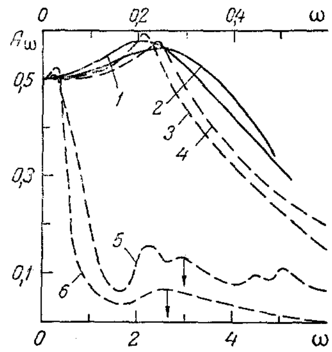

Fig. 7

Fig. 6. The change in the OF radius with a stepped perturbation of the draw rate (solution of the linearized system of equations, $\tilde{V}_b = 1\%$): 1) $T_{p2} = 2250°C$, $V_b = 0.5$ m/sec; 2) 2150; 3; 3) 2250; 3; 4) perturbation F_f for 5; 5) 2350; 10 ($\tilde{R}_k$ should be replaced by $\tilde{R}_b$).

Fig. 7. The AFC of the nonisothermal regime of OF formation (peturbation of the draw rate): 1) $T_{p2} = 2250°C$, $V_b = 0.5$ m/sec; 2) 2250; 3; 3) 2250; 0.5; 4) 2250; 3; 5) 2250; 10; 6) 2150; 10 (the solid lines are the solution of Eqs. (6)-(11) and the broken lines are the Fourier transform of the weighting function).

NOTATION

T, temperature; τ, time; λ, dimensionless effective thermal conductivity of the melt; x, longitudinal coordinate; R, radius of the jet; V, velocity of the melt; T_p, furnace temperature; $\tilde{y}$, perturbation of the function y, y, $y \in \{R, V, T, St, T_p, V_0, R_0, T_k, V_b, F_f\}$; $\bar{y}$, expansion of the function for performing the linearization $\bar{y} = y(1+\tilde{y})$, $y \in \{R, V, T, St, T_p, V_0, R_0, V_b, T_k, F_f\}$; φ_i, α_i, β_i, coefficients in the linearized equations; η, coordinate along the surface of the furnace; T_0, temperature of the gas; R_p, radius of the furnace; ε_p, emissivity of the furnace; β, absorption coefficient; k_i, weighting factors; V_0, feed rate; R_0, radius of the preform; T_k, temperature of the melt at x = 0; V_b, draw rate; T_{p1}, T_{p2}, minimum and maximum temperatures of the furnace; ξ, parameter in the functional $T_p(\eta)$; θ_0, length of the heated section; F_f, tension force; μ, dimensionless viscosity of the melt; a_2, a parameter in the dependence μ(T); $Pe = \rho V_0 c l/\lambda_T$, Peclet number, where ρ, c, and λ_T, density, heat capacity, and molecular conductivity of the melt; ℓ, length of the computing region; $St = h/\rho V_0 \ell$, Stanton number; h, external heat transfer coefficient; $Re = \rho V_0 l/\mu_0$, Reynolds number; μ_0, velocity scale; $\chi = n_c^2 \sigma_0 T_0^4 / c\rho V_0$, where n_c is the refractive index of the gas blown through the heating zone; σ_0, Stefan–Boltzmann constant; $We = V_0 \rho l/\sigma$, Weber's number; σ, surface tension; $Fr = 2V_0/g\ell$, Froude's number; g, acceleration of gravity; $W = V_b/V_0$, velocity factor; $(...)' = d/dx$; ΔR_b, difference between the stationary value of R_b and the instantaneous value; $(...)'' = d^2/dx^2$.

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